EPFL

Probability and Statistics for IC
2023, Fall semester

Probability and Statistics: Midterm solutions

Exercise 1. At the end of a video game, the player must kill a monster sampled randomly as follows:

- 1 time out of 10, it is a *dragon*,
- 3 times out of 10, it is a *troll*,
- the rest of the time, it is a *giant*.

When the monster dies, the player gets a chance to get a *ruby*:

- the dragon always gives a ruby,
- the troll gives a ruby 1 time out of 2,
- the giant gives a ruby 1 time out of 5.

We assume that the game is very easy; thus the players always succeed at killing the monster. Different rounds of the game are independent.
Alice plays the game. We denote D (respectively T, G) the event "the monster is a dragon" (respectively a troll, a giant). We denote R the event "Alice wins a ruby", and $p = \mathbb{P}(R)$.

1. Compute p.

2. Alice won a ruby! What is the probability that the monster was a troll?

3. Bob decides to play 6 games. We denote S the number of rubies that he gets.

- **(a)** What is the distribution of S?
- **(b)** What is the expectation of S?
- **(c)** What is the probability that Bob wins exactly 3 rubies?
- **(d)** What is the probability that Bob wins at least 1 ruby?

4. Charlie decides to play until he wins one ruby. We denote X the number of games Charlie plays.

- **(a)** What is the distribution of X?
- **(b)** What is the expectation of X?
- **(c)** What is the probability that Charlies does exactly 3 games?

Solution. (7.5 points)

1. (2pt) Using the law of total probability we write,

$$\begin{aligned} p = \mathbb{P}(R) &= \mathbb{P}(R|D)\mathbb{P}(D) + \mathbb{P}(R|T)\mathbb{P}(T) + \mathbb{P}(R|G)\mathbb{P}(G) \\ &= 1 \cdot \frac{1}{10} + \frac{1}{2} \cdot \frac{3}{10} + \frac{1}{5} \cdot \frac{6}{10} \\ &= \frac{1}{10} \cdot \frac{10 + 15 + 12}{10} \\ &= \frac{37}{100}. \end{aligned}$$

Grading: 1 point for the correct answer and 1 for writing the way to compute it (e.g., the law of total probability).

2. (1pt) Using Bayes rule,

$$\mathbb{P}(T|R) = \frac{\mathbb{P}(R|T)\mathbb{P}(T)}{p} = \frac{1}{2} \cdot \frac{3}{10} \cdot \frac{100}{37} = \frac{15}{37}.$$

Grading: 0.5 for the correct answer and 0.5 for writing the way to compute it (e.g., Bayes-rule).

3. (2.5pt)

   a) (1pt) Since S is the sum of 6 independent events, with probability of success p, it follows that $S \sim \text{Binomial}(p, 6)$ (Grading: 0.5 if the answer is correct and 0.5 if the explanation is given)

   b) (0.5pt) $\mathbb{E}[S] = 6p$ (Grading: 0.5 if answer is correct)

   c) (0.5pt) $\mathbb{P}(S = 3) = \binom{6}{3} p^3 (1-p)^3$ (Grading: 0.5 if the answer is correct)

   d) (0.5pt) $\mathbb{P}(S \geq 1) = 1 - \mathbb{P}(S = 0) = 1 - (1-p)^6$ (Grading: 0.5 if the answer is correct)

4. (2pt)

   a) (1pt) Since X denotes the number of trials to get the first success of independent events of probability p, it follows that $X \sim \text{Geom}(p)$ (Grading: 0.5 if the answer is correct and 0.5 if the explanation is given)

   b) (0.5pt) $\mathbb{E}[X] = 1/p = 100/37$ (Grading: 0.5 if the answer is correct)

   c) (0.5pt) $\mathbb{P}(X = 3) = (1-p)^2 p$. (Grading: 0.5 if the answer is correct)

□

Exercise 2. Let (X, Y) be a joint random variable whose probability mass function is given by the following table:

$X \backslash Y$	-1	1
0	0.2	0.4
1	0.3	0.1

1. What are the marginal distributions of X and Y?

2. Are X and Y independent? (As always, justify your answer.)

2. Compute $\mathbb{E}[X]$, $\mathbb{E}[Y]$, $\mathbb{E}[X+Y]$ and $\mathbb{E}[XY]$.

Solution: (4 points)

1. (1pt) Note that $f_Y(y) = f(x = 0, y) + f(x = 1, y)$, as a result

$$f_Y(1) = 0.4 + 0.1 = 0.5 \qquad f_Y(-1) = 0.2 + 0.3 = 0.5. \tag{1}$$

Similarly, we have $f_X(x) = f(x, y = -1) + f(x, y = +1)$. Thus,

$$f_X(x) = \begin{cases} 0.6 & \text{if } x = 0\ , \\ 0.4 & \text{if } x = 1 \end{cases}. \tag{2}$$

Grading: 0.5 for each of the marginal distributions. If the marginals are correct but no explanation is given, 0.5 would be removed.

2. (1pt) No. For instance,

$$f_{X,Y}(0, -1) = 0.2 \tag{3}$$
$$f_X(0) \cdot f_Y(-1) = 0.6 \cdot 0.5 = 0.3. \tag{4}$$

Since, $f_{X,Y}(0, -1) \neq f_X(0) \cdot f_Y(-1)$, it follows that X and Y are not independent.
Grading: 0.5 for the correct answer and 0.5 for the explanation

3. (2pt)

$$\mathbb{E}[X] = 0.4 \cdot 1 = 0.4 \tag{5}$$
$$\mathbb{E}[Y] = 0.5 \cdot 1 + 0.5 \cdot -1 = 0 \tag{6}$$
$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 0.4 \tag{7}$$
$$\mathbb{E}[X \cdot Y] = \sum_{x=0,1} \sum_{y=-1,1} xy f_{(X,Y)}(x,y) = -1 \cdot 0.3 + 1 \cdot 0.1 = -0.2. \tag{8}$$

Grading: 0.5 for each of the correctly computed expectations

□

Exercise 3. We consider 5 urns: 3 of type A and 2 of type B. Each urn of type A contains 1 red ball and 3 white balls; each urn of type B contains 2 red balls and 2 white balls. We sample an urn uniformly at random; then, in the chosen urn, we sample two balls uniformly at random, *with replacement* (i.e., we replace the first sampled ball back in the urn before sampling the second ball).
Let U_A (resp. U_B) denote the event "the chosen urn is of type A (resp. of type B)". Let X denote number of red balls among the sampled balls.

1. Compute $\mathbb{P}(U_A)$, $\mathbb{P}(U_B)$ and $\mathbb{P}(X = k|U_B)$ for $k = 0, 1, 2$.

2. What is the probability mass function of X?

3. Compute $\mathbb{E}[X]$.

4. Knowing that we have sampled a red ball and a white ball, what is the probability that we have sampled an urn of type A?

5. Are the events $\{X = 1\}$ and U_A independent? As always, justify your answer.

Solution: (6 points)

1. (1.5 pt)

$$\mathbb{P}(U_A) = \frac{3}{5}, \quad \mathbb{P}(U_B) = \frac{2}{5}$$
$$\mathbb{P}(X = k|U_B) = \binom{2}{k} \cdot \frac{1}{4} \text{ (Note that } X|U_B \text{ follows a Binomial distribution with } (n,p) = (2, 1/2))$$

Grading: 0.5 for the correct values on the first line, 0.5 for correct values of $\mathbb{P}(X = k|U_B)$ and 0.5 for its explanation.

2. (1.5 pt) For $k = 0, 1, 2$,

$$\begin{aligned} \mathbb{P}(X = k) &= \mathbb{P}(X = k|U_A)\mathbb{P}(U_A) + \mathbb{P}(X = k|U_B)\mathbb{P}(U_B) \\ &= \frac{3}{5}\binom{2}{k}\left(\frac{1}{4}\right)^k \left(\frac{3}{4}\right)^{2-k} + \frac{2}{5}\binom{2}{k}\frac{1}{4} \end{aligned}$$

Grading: 0.5 for correct values (or correct expression), 1 for the correct explanation.

3. (1 pt) One can observe that $X|U_A \sim \text{Bin}(1/4, 2)$ and $X|U_B \sim \text{Bin}(1/2, 2)$. Thus, $\mathbb{E}[X|U_A] = 1/2$ and $\mathbb{E}[X|U_B] = 1$. We can compute

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[X|U_A] \cdot \mathbb{P}(U_A) + \mathbb{E}[X|U_B] \cdot \mathbb{P}(U_B) \\ &= \frac{1}{2} \cdot \frac{3}{5} + 1 \cdot \frac{2}{5} = \frac{7}{10}. \end{aligned}$$

Grading: 0.5 for the correct answer and 0.5 for explanation

4. (1 pt) We first compute

$$\begin{aligned}\mathbb{P}(X=1) &= \mathbb{P}(X=1|U_A)\mathbb{P}(U_A)+\mathbb{P}(X=1|U_B)\mathbb{P}(U_B)\\ &= 2\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{3}{5}+2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{5}=\frac{17}{40}\end{aligned}$$

We then apply Bayes-rule

$$\begin{aligned}\mathbb{P}(U_A|X=1) &= \frac{\mathbb{P}(X=1|U_A)\cdot\mathbb{P}(U_A)}{\mathbb{P}(X=1)}\\ &= \frac{3/8\cdot 3/5}{17/40}=\frac{9}{17}\end{aligned}$$

Grading: 0.5 for the correct answer and 0.5 for explanation

5. (1 pt) No. In fact, $\mathbb{P}(X=1)\neq\mathbb{P}(X=1|U_A)$.
Grading: 0.5 for the correct answer and 0.5 for giving an example.

□

Exercise 4 (Multiple Choice Question; +3 if the correct and −1 if a wrong choice is selected.)**.** Let $X\sim\text{Uniform}(1,3)$ and Y such that $Y|X\sim\exp(X)$, i.e., $f_{Y|X}(y|x)=xe^{-yx}$ for $y\in(0,\infty), x\in(1,3)$. What is the value of $\mathbb{E}[X^2Y]$?

(a) 1

(b) 2

(c) 4

(d) 9

(The above (and only the above) question is a multiple choice question, you do not need to deliver a solution.)

Solution: Note that

$$\mathbb{E}[X^2Y]=\mathbb{E}_X[\mathbb{E}_Y[X^2Y|X]]=\mathbb{E}_X[X^2\mathbb{E}_Y[Y|X]]=\mathbb{E}_X[X]=\frac{1+3}{2}=2.$$

We used the fact that $Y|X\sim\exp(X)$ and thus $E_Y[Y|X]=\frac{1}{X}$. One can also solve this problem by computing the integral below:

$$\int x^2yf(x,y)dxdy=\int_1^3\int_0^\infty x^2yf(y|x)f(x)dydx=\int_1^3\int_0^\infty x^2y\frac{1}{2}xe^{-yx}dydx$$

□

Exercise 5. Let $F\colon\mathbb{R}\to[0,1]$ be the function defined by

$$F(x)=\begin{cases}0 & \text{if } x<0\,,\\ x^2 & \text{if } 0\le x\le 1\,,\\ 1 & \text{if } 1<x\,.\end{cases}$$

1. Justify that F is a cumulative distribution function.

2. Justify that the law associated to F is continuous.

3. Compute the density f associated to this law.

4. Let X be a random variable with cumulative distribution function F.

(a) Compute $\mathbb{P}(X > 1/2)$.

(b) Compute $\mathbb{E}[X]$.

(c) Compute the variance of X.

Solution. (5.5 points)

1. (1.5pt) One can verify that F is such that
   - $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$;
   - F is increasing;
   - F is continuous on the right.

   Thus, F is a valid CDF.
   Grading: 0.5 for each of the points above

2. (0.5pt) Since $F(x)$ is continuous, it follows that the law associated to F is continuous.

3. (1pt)

$$f(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ 2x & \text{if } 0 \leq x \leq 1\,, \\ 0 & \text{if } x > 1\,. \end{cases} \tag{9}$$

   Grading: 0.5 for differentiation, 0.5 for the correct values, including the zones that PDF is 0.

4. (2.5pt) One can compute that:
   a) (0.5pt) $\mathbb{P}(X > 1/2) = 3/4$
   b) (1pt) $\mathbb{E}[X] = \int_0^1 xf(x)dx = \int_0^1 2x^2 dx = 2/3$
   c) (1pt) $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \int_0^1 x^2 f(x)dx - 4/9 = \int_0^1 2x^3 dx - 4/9 = 1/2 - 4/9 = 1/18$.

   Grading: 0.5 for each correctly computed value; for parts b) and c) 0.5 each for the integrations.

□

Exercise 6. Let $(X, Y) \in \mathbb{R}^2$ be a random point, sampled uniformly in the unit disk. Said differently, (X, Y) has density

$$f_{(X,Y)}(x, y) = \frac{1}{\pi} I(x^2 + y^2 \leq 1) = \begin{cases} \frac{1}{\pi} & \text{if } x^2 + y^2 \leq 1\,, \\ 0 & \text{if } x^2 + y^2 > 1\,. \end{cases}$$

1. What is the density of X?

2. Are X and Y independent? (As always, justify your answer.)

Solution: (2.5 points)

1. (1.5 pt)

$$\begin{aligned} f_X(x) &= \int_{\mathbb{R}} f_{(X,Y)}(x, y)dy \\ &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy \\ &= \frac{2}{\pi}\sqrt{1 - x^2} \end{aligned}$$

   Grading: 1.0 for the integration with correct bounds and 0.5 for the final correct answer.
   Note: Another correct approach is the graphical solution yielding $\frac{2}{\pi} \sin(\arccos(x))$.

2. (1 pt) No. In fact, by symmetry, $f_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}$. From the marginal density functions, we see that $\mathbb{P}(X > 1/\sqrt{2}) = \mathbb{P}(Y > 1/\sqrt{2}) > 0$. However,

$$\mathbb{P}(X > 1/\sqrt{2}, Y > 1/\sqrt{2}) \leq \mathbb{P}(X^2 + Y^2 > 1) = 0\,.$$

Thus

$$0 = \mathbb{P}(X > 1/\sqrt{2}, Y > 1/\sqrt{2}) \neq \mathbb{P}(X > 1/\sqrt{2})\mathbb{P}(Y > 1/\sqrt{2}) > 0\,.$$

Thus X and Y are not independent.
Grading: 0.5 for the correct answer and 0.5 for explanation

□

Exercise 7. *This exercise is significantly harder than the previous ones, but not worth many points. Try it once you have finished the rest of the midterm.*
We remind that:

- *an exponential random variable with rate λ has density $f(x) = \lambda e^{-\lambda x} I(x > 0)$,*
- *a gamma random variable has with shape parameter α and rate λ has density*

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I(x > 0)\,,$$

- *if n is a non-negative integer, $\Gamma(n+1) = n!$, and*
- *Poisson random variable with rate λ has probability mass function $f(n) = e^{-\lambda}\lambda^n/n!$, $n = 0, 1, \ldots$.*

Let $X_1, X_2, X_3, \ldots$ be an infinite sequence of independent random variables of exponential law with rate parameter 1. We denote $T_0 = 0$ and for all $n \geq 1$, $T_n = X_1 + \cdots + X_n$. For all $t \geq 0$, we denote $N_t = \max\{n \geq 0 \,|\, T_n \leq t\}$.

1. (a) Compute the cumulative distribution function of T_2. Deduce that T_2 is a gamma random variable with shape parameter 2 and rate 1. (Like everywhere else, a proof is required.)

(b) Let $n \geq 1$. Using the same method, compute the law of T_n.

2. Let $t > 0$. Compute the law of N_t.

3. Let $n \geq 1$.

(a) Compute the joint law of $(T_1, \ldots, T_n)$.

(b) Compute the conditional law of $(T_1, \ldots, T_n)$ given that $N_t = t$.

Solution. (6 points)

1. (2pt)

a) (1pt) The CDF of T_2 can be computed as follows:

$$\begin{aligned}
F_{T_2}(t) = \mathbb{P}(X_1 + X_2 \leq t) &= \int_0^t dx_1 \int_0^{t-x_1} dx_2 f_{(X_1,X_2)}(x_1, x_2) && (10)\\
&\overset{\text{(indep.)}}{=} \int_0^t dx_1 \int_0^{t-x_1} dx_2 f_{X_1}(x_1) f_{X_2}(x_2) && (11)\\
&= \int_0^t dx_1 e^{-x_1} \int_0^{t-x_1} dx_2 e^{-x_2} && (12)\\
&= \int_0^t dx_1 e^{-x_1}(1 - e^{-(t-x_1)}) && (13)\\
&= \int_0^t dx_1 (e^{-x_1} - e^{-t}) && (14)\\
&= 1 - e^{-t}(t+1). && (15)
\end{aligned}$$

Differentiating the above with respect to t, we obtain

$$\frac{d}{dt}F_{T_2}(t) = -e^{-t} + e^{-t}(t+1) = te^{-t} \tag{16}$$

that is the PDF of a Gamma(2,1) random variable.

b) (1pt) We show by induction that $T_n \sim \text{Gamma}(n, 1)$. The base case is given by point a). The inductive step is as follows: assume that $T_n \sim \text{Gamma}(n, 1)$, then

$$F_{T_{n+1}}(t) = \mathbb{P}(T_n + X_{n+1} \leq t) = \int_0^t dy \int_0^{t-y} dx f_{(T_n, X_{n+1})}(y, x) \tag{17}$$

$$= \int_0^t dy \int_0^{t-y} dx f_{T_n}(y) f_{X_{n+1}}(x) \tag{18}$$

$$= \int_0^t dy \frac{1}{(n-1)!} y^{n-1} e^{-y} \int_0^{t-y} dx e^{-x} \tag{19}$$

$$= \int_0^t dy \frac{1}{(n-1)!} y^{n-1} e^{-y} (1 - e^{-(t-y)}) \tag{20}$$

$$= \int_0^t dy \frac{1}{(n-1)!} y^{n-1} e^{-y} - e^{-t} \int_0^t dy \frac{1}{(n-1)!} y^{n-1}. \tag{21}$$

Differentiating the above with respect to t, we obtain

$$\frac{d}{dt}F_{T_{n+1}}(t) = \frac{1}{(n-1)!} t^{n-1} e^{-t} - e^{-t} \frac{1}{(n-1)!} t^{n-1} + e^{-t} \int_0^t dy \frac{1}{(n-1)!} y^{n-1} \tag{22}$$

$$= \frac{1}{n!} t^n e^{-t}, \tag{23}$$

that is the PDF of a Gamma distribution with shape parameter $n+1$ and scale parameter 1.

Grading: 1 point each part, writing the correct integrals with the correct bounds has 0.5.

2. (2pt) For all $n \in \mathbb{N}$, we have $\mathbb{P}(N_t = n) = \mathbb{P}(T_n \leq t, T_{n+1} > t)$, since the sequence $(T_n)_{n \geq 0}$ is increasing. Note that T_n and T_{n+1} are not independent, since they both depend on $X_1, \ldots, X_n$. However, T_n and X_{n+1} are independent. Thus, for $n \geq 1$, we get

$$\mathbb{P}(N_t = n) = \mathbb{P}(T_n \leq t, T_n + X_{n+1} > t) \tag{24}$$

$$= \mathbb{P}(T_n \leq t, X_{n+1} > t - T_n) \tag{25}$$

$$= \int_0^t dy \int_{t-y}^{\infty} dx f_{(T_n, X_{n+1})}(y, x) \tag{26}$$

$$= \int_0^t dy f_{T_n}(y) \int_{t-y}^{\infty} dx f_{X_{n+1}}(x) \tag{27}$$

$$= \int_0^t dy \frac{1}{(n-1)!} y^{n-1} e^{-y} \int_{t-y}^{\infty} dx e^{-x} \tag{28}$$

$$= \int_0^t dy \frac{1}{(n-1)!} y^{n-1} e^{-y} e^{-(t-y)} \tag{29}$$

$$= e^{-t} \int_0^t dy \frac{1}{(n-1)!} y^{n-1} = \frac{1}{n!} t^n e^{-t}, \tag{30}$$

thus $N_t \sim \text{Poisson}(t)$.
Grading: 0.5 for the correct answer, 0.5 for $\mathbb{P}(N_t = n) = \mathbb{P}(T_n \leq t, T_n + X_{n+1} > t)$, and 1 for the correct computation.
Note that $\mathbb{P}(N_t = n) = \mathbb{P}(T_n \leq t, T_{n+1} > t) = \mathbb{P}(T_n \leq t) - \mathbb{P}(T_n \leq t, T_{n+1} \leq t) = \mathbb{P}(T_n \leq t) - \mathbb{P}(T_{n+1} \leq t)$ can also lead to an alternative solution (one has to use integration by parts).

3. (2pt)

a) (1pt) First, consider the case where $n = 2$.

$$f_{(T_1,T_2)}(t_1,t_2) = f_{(X_1,X_1+X_2)}(t_1,t_2) \tag{31}$$
$$= f_{X_1}(t_1) f_{X_1+X_2|X_1}(t_2|t_1) \tag{32}$$
$$= f_{X_1}(t_1) f_{X_2|X_1}(t_2 - t_1|t_1). \tag{33}$$

Since X_1 and X_2 are independent we can write $f_{X_2|X_1}(t_2 - t_1|t_1) = f_{X_2}(t_2 - t_1)$, and plug in the density functions of X_1 and X_2

$$f_{(T_1,T_2)}(t_1,t_2) = f_{X_1}(t_1) f_{X_2}(t_2 - t_1) \tag{34}$$
$$= e^{-t_1}\mathbb{1}(t_1 > 0)e^{-(t_2-t_1)}\mathbb{1}(t_2 > t_1) \tag{35}$$
$$= e^{-t_2}\mathbb{1}(0 < t_1 < t_2). \tag{36}$$

We now prove by induction that for general n,

$$f_{(T_1,...,T_n)}(t_1,...,t_n) = e^{-t_n}\mathbb{1}(0 < t_1 < t_2 < ... < t_n) \tag{37}$$

The base case is given by the above. For the inductive step, assume that (37) holds for n. Then,

$$f_{(T_1,...,T_n,T_{n+1})}(t_1,...,t_n,t_{n+1}) = f_{(T_1,...,T_n,T_n+X_{n+1})}(t_1,...,t_n,t_{n+1}) \tag{38}$$
$$= f_{(T_1,...,T_n)}(t_1,...,t_n) f_{T_n+X_{n+1}|(T_1,...,T_n)}(t_{n+1}|t_1,...,t_n) \tag{39}$$
$$= f_{(T_1,...,T_n)}(t_1,...,t_n) f_{X_{n+1}|(T_1,...,T_n)}(t_{n+1} - t_n|t_1,...,t_n). \tag{40}$$

Observe one more time that X_{n+1} is independent from $T_1,...,T_n$, which gives

$$f_{(T_1,...,T_n,T_{n+1})}(t_1,...,t_n,t_{n+1}) = f_{(T_1,...,T_n)}(t_1,...,t_n) f_{X_{n+1}}(t_{n+1} - t_n) \tag{41}$$
$$= e^{-t_n}\mathbb{1}(0 < t_1 < t_2 < ... < t_n)e^{-(t_{n+1}-t_n)}\mathbb{1}(t_{n+1} > t_n) \tag{42}$$
$$= e^{-t_{n+1}}\mathbb{1}(0 < t_1 < t_2 < ... < t_{n+1}), \tag{43}$$

which concludes the proof.

b) (1pt) Recall that $N_t = n \iff T_n \le t, T_{n+1} > t$. We first compute the joint distribution of $(T_1,...,T_n,N_t)$:

$$f_{(T_1,...,T_n,N_t)}(t_1,...t_n,n) = \int_0^\infty dt_{n+1} f_{(T_1,...,T_n,T_{n+1},N_t)}(t_1,...t_n,t_{n+1},n) \tag{44}$$
$$= \int_0^\infty dt_{n+1} f_{(T_1,...,T_n,T_{n+1})}(t_1,...t_n,t_{n+1})\mathbb{1}(t_n \le t, t_{n+1} > t) \tag{45}$$
$$= \int_0^\infty dt_{n+1} e^{-t_{n+1}}\mathbb{1}(0 < t_1 < ... < t_{n+1})\mathbb{1}(t_n \le t, t_{n+1} > t) \tag{46}$$
$$= \int_t^\infty dt_{n+1} e^{-t_{n+1}}\mathbb{1}(0 < t_1 < ... < t_n \le t) \tag{47}$$
$$= e^{-t}\mathbb{1}(0 < t_1 < t_2 < ... < t_n \le t). \tag{48}$$

Applying the formula for the conditional probability we get

$$\begin{aligned} f_{(T_1,...,T_n)|N_t}(t_1,...,t_n|n) &= \frac{f_{(T_1,...,T_n,N_t)}(t_1,...,t_n,n)}{\mathbb{P}(N_t = n)} \\ &= \frac{e^{-t}\mathbb{1}(0 < t_1 < t_2 < ... < t_n \le t)}{e^{-t}t^n/n!} \\ &= n!t^{-n}\mathbb{1}(0 < t_1 < t_2 < ... < t_n \le t). \end{aligned}$$

Grading: for each part, 0.5 for the correct final answer and 0.5 for the correct computation

□